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# Switched Controller Order Reduction

Hamid Reza Shaker and Rafael Wisniewski

**Abstract**—A method for order reduction of switched controllers is presented in this paper. The proposed technique is based on the generalized gramian framework for model reduction and it is the carry-over of the method in [20] and [9]. To the best of our knowledge, there is no other reported result on switched controller reduction in the literature. The method is an LMI-based technique in which to avoid numerical instability and also to increase the numerical efficiency, generalized gramian based Petrov-Galerkin projection is constructed instead of the similarity transform approach for reduction. The stability of the closed-loop system under arbitrary switching signal is proven to be preserved in the reduction and the technique is applicable to both continuous and discrete time systems. The performance of the proposed method is illustrated by numerical example.

## I. INTRODUCTION

THE ever-increasing need for accurate mathematical modeling of physical as well as artificial processes for simulation and control leads to models and controllers of high complexity. This problem demands efficient computational prototyping tools to replace such complex models/controllers by an approximate simpler model/controller, which are capable of capturing dynamical behavior and preserving essential properties of the complex one. Due to this fact model reduction methods have become increasingly popular over the last two decades [1],[2],[3]. Such methods are designed to extract a reduced order state space model that adequately describes the behavior of the system in question.

In particular, there are several reasons for the preference of low order controllers rather than higher order ones which usually appear when dealing with practical large-scale models. A low-order controller for a large scale system brings ease of implementation. As opposed to a high-order controller that might require expensive or complicated hardware; the low-order controller requires less complicated and more easily available hardware to implement, to fix and to understand. Low-order controllers are computationally less demanding. The effects of computational delay due to the complexity of the controller on the stability and performance of the closed loop decrease when low order controllers are used instead of the one of high order. Therefore the problem of controller order reduction gained considerable attention in recent years [7]-[10]. On the other hand, most of the methods that are proposed so far for control and analysis of hybrid and switched systems suffer from high computational burden when dealing with

large-scale dynamical systems. Because of the weakness of nonlinear model reduction techniques and also pressing needs for efficient analysis and control of large-scale dynamical hybrid and switched systems, model reduction draws lately attentions of the hybrid systems research community [15]-[24]. Some works have been focused on ordinary model reduction methods that have potential applications in modeling and analysis of hybrid systems [15]-[19] motivated by reachability analysis and safety verification problem. Some researches address the problem of model reduction of switched and hybrid systems directly [20]-[24]. Among those, the generalized grammian framework for model reduction of switched systems is proposed in [20] which gives the base for our method for switched controller reduction in this paper.

This method can be categorized as SVD based model reduction methods. Balanced model reduction is one of the most common SVD-based model reduction schemes. It was presented in [4] for the first time.

To apply balanced reduction, first the system is represented in a basis, where the states, which are difficult to reach are simultaneously difficult to observe. This is achieved by simultaneously diagonalizing the reachability and the observability gramians, which are solutions to the reachability and the observability Lyapunov equations. Then, the reduced model is obtained by truncating the states which have this property. Balanced model reduction method is modified and developed from different viewpoints [1],[2]. Generalized grammian based reduction method is one of the technique that are developed based on balanced model reduction which uses the generalized gramians instead of gramians[5]. In this method in order to compute the generalized gramians, one solves Lyapunov inequalities instead of Lyapunov equations. This method is also used to devise a technique for structure preserving model reduction methods in [6].

To the best of our knowledge, there is no reported result on switched controller reduction in literature. In this paper we propose a technique for switched controller reduction which can be considered as a carry-over of the method in [20] and [9]. The method is an LMI-based technique. We modified the original method in [5] to avoid numerical instability and also to achieve more numerical efficiency by building Petrov-Galerkin projection based on generalized gramians. We generalized the framework to switched controller reduction by solving systems of Lyapunov inequalities to find common generalized gramians.

The paper is organized as follows: In the next section we review balanced reduction method and balanced reduction

technique based on generalized gramian. Section III presents how generalized gramian based method can be applied to switched controller reduction problem followed by some remarks on numerical implementation of the algorithm and using projection for generalized gramian based reduction method is suggested instead of balancing and truncation. Section III ends up with a brief discussion on stability, feasibility and an error bound. Section IV presents our numerical results.

The notation used in this paper is as follows:  $M^*$  denotes transpose of matrix if  $M \in \mathbb{R}^{n \times m}$  and complex conjugate transpose if  $M \in \mathbb{C}^{n \times m}$ . The norm  $\|\cdot\|_\infty$  denotes the  $H_\infty$  norm of a rational transfer function. The standard notation  $>, \geq (<, \leq)$  is used to denote the positive (negative) definite and semidefinite ordering of matrices.

## II. BALANCED TRUNCATION, GENERALIZED GRAMIANs AND MODEL REDUCTION OF SWITCHED SYSTEMS

Balanced truncation is a well-known method for model reduction of dynamical systems, see for example [1],[2]. The basic approach relies on balancing the gramians of the systems. For dynamical systems with minimal realization:

$$G(s) := (A, B, C, D) \quad (1)$$

where  $G(s)$  is transfer matrix with associated state-space representation:

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(t) \in \mathbb{R}^n \\ y(t) = Cx(t) + Du(t) \end{cases} \quad (2)$$

gramians are given by the solutions of the Lyapunov equations:

$$\begin{aligned} AP + PA^* + BB^* &= 0 \\ A^*Q + QA + C^*C &= 0 \end{aligned} \quad (3)$$

For stable  $A$ , they have a unique positive definite solutions  $P$  and  $Q$ , called the controllability and observability gramians. In balanced reduction, first the system is transformed to the balanced structure in which gramians are equal and diagonal:

$$\begin{aligned} P = Q &= \text{diag}(\sigma_1 I_{k_1}, \dots, \sigma_q I_{k_q}) \\ \sum_{j=1}^q k_j &= n \end{aligned} \quad (4)$$

where  $\sigma_i > \sigma_{i+1}$  and they are called Hankel singular values.

The reduced model can be easily obtained by truncating the states which are associated with the set of the least Hankel singular values. Applying the method to stable, minimal realization  $G(s)$ , If we keep all the states associated to  $\sigma_m$  ( $1 \leq m \leq r$ ), by truncating the rest, the reduced model  $G_r(s)$  will be minimal and stable and satisfies[1][2]:

$$\|G(s) - G_r(s)\|_\infty \leq 2 \sum_{j=r+1}^q \sigma_j \quad (5)$$

One of the closely related model reduction methods to the

balanced truncation is balanced reduction based on generalized gramian that is presented in [5]. The following Proposition allows us to develop generalized version of different gramian based reduction methods as we did in [20] for frequency domain balanced reduction within frequency bound.

**Proposition 1:** Suppose  $A$  is stable and  $X$  is a solution of Lyapunov equation:

$$A^*X + XA + Q = 0 \quad (6)$$

where  $Q \geq 0$ . If a symmetric  $X_g$  satisfies:

$$A^*X_g + X_gA + Q \leq 0 \quad (7)$$

Then:  $X_g \geq X$ .

**Proof:** see [5] or [20].

In generalized gramian based balanced method, instead of Lyapunov equations (3), the following Lyapunov inequalities should be solved:

$$\begin{aligned} AP_g + P_gA^* + BB^* &\leq 0 \\ A^*Q_g + Q_gA + C^*C &\leq 0 \end{aligned} \quad (8)$$

For stable  $A$ , they have positive definite solutions  $P_g$  and  $Q_g$ , called the generalized controllability and observability gramians. Note that these gramians are not unique. The rest of this model reduction method is the same as the aforementioned balanced truncation method, the only difference is that in this algorithm the balancing and truncation are based on generalized gramian instead of ordinary gramian. In this method we have generalized Hankel singular values ( $\gamma_i$ ) which are the diagonal elements of balanced generalized gramians instead of Hankel singular values  $\sigma_i$  which are the diagonal elements of balanced ordinary gramians. For the error bound also the same result holds but in terms of the generalized Hankel singular values instead of Hankel singular values. It is worth to mention that,  $\gamma_i \geq \sigma_i$ . Therefore the error bound in balanced reduction based on generalized gramian is greater equal than the error bound in ordinary balanced model reduction.

In order to develop this technique for model reduction of switched systems one solves a system of Lyapunov LMI to find common observability/controllability generalized gramian of subsystems and build balancing transformation based on these generalized gramians[20].

This reduction framework is stability preserving. In other words, the reduced order switched system is stable under arbitrary switching signal [20]. In the next section we try to develop this method for switched controller reduction.

## III. SWITCHED CONTROLLER REDUCTION METHOD

In this section we present a method for switched controller reduction followed by a brief discussion on modifications in numerical implementation of the algorithm. In the last subsection we elaborate more on the method

studying stability, feasibility and approximation error.

#### A. Generalized Gramian Framework for Switched Controller Reduction

One of the most important subclasses of hybrid systems are Linear switched systems. Linear switched system is a dynamical system specified by the following equations:

$$\Sigma: \begin{cases} \dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) = C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t) \end{cases} \quad (9)$$

where  $x(t) \in \mathbb{R}^n$  is the continuous state,  $y(t) \in \mathbb{R}^p$  is the continuous output,  $u(t) \in \mathbb{R}^m$  is the measurable input, and  $\sigma: \mathbb{R}^{\geq 0} \rightarrow K \subset \mathbb{N}$  is the switching signal that is a piecewise constant map of the time.  $K$  is the set of discrete modes, and it is assumed to that it is finite. For each  $i \in K$ ,  $A_i, B_i, C_i, D_i$  are matrices of appropriate dimensions.

Consider a general switched system with switched controller in the following closed loop configuration (see Fig. 1.)

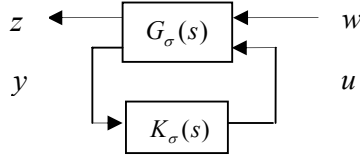


Fig. 1. Closed loop Switched System with Switching Controller

In this configuration  $G_{\sigma}(s) := (A_{\sigma}, B_{\sigma}, C_{\sigma}, D_{\sigma})$  is  $n$ -th order switched system and  $K_{\sigma}(s) := (A_{k\sigma}, B_{k\sigma}, C_{k\sigma}, D_{k\sigma})$  is  $m$ -th order switched controller. The transfer matrix from  $w$  to  $z$  is:

$$T_{zw} := (\bar{A}_{\sigma}, \bar{B}_{\sigma}, \bar{C}_{\sigma}, \bar{D}_{\sigma}) \quad (10)$$

where:

$$\bar{A}_{\sigma} = \begin{bmatrix} A_{\sigma} + B_{2\sigma}L_{\sigma}D_{k\sigma}C_{2\sigma} & B_{2\sigma}L_{\sigma}C_{k\sigma} \\ B_{k\sigma}F_{\sigma}C_{2\sigma} & A_{k\sigma} + B_{k\sigma}F_{\sigma}D_{22\sigma}C_{k\sigma} \end{bmatrix} \quad (11)$$

$$\bar{B}_{\sigma} = \begin{bmatrix} B_{1\sigma} + B_{2\sigma}L_{\sigma}D_{k\sigma}D_{21\sigma} \\ B_{k\sigma}F_{\sigma}D_{21\sigma} \end{bmatrix} \quad (12)$$

$$\bar{C}_{\sigma} = [C_{1\sigma} + D_{12\sigma}D_{k\sigma}F_{\sigma}C_{2\sigma} \quad D_{12\sigma}L_{\sigma}C_{k\sigma}] \quad (13)$$

$$\bar{D}_{\sigma} = D_{11\sigma} + D_{12\sigma}D_{k\sigma}F_{\sigma}D_{21\sigma} \quad (14)$$

$$L_{\sigma} = (I - D_{k\sigma}D_{22\sigma})^{-1}, F_{\sigma} = (I - D_{22\sigma}D_{k\sigma})^{-1} \quad (15)$$

Note that in the above configuration switched plant  $G_{\sigma}(s)$  is partitioned where inputs to  $B_{1\sigma}$  are the

disturbances, inputs to  $B_{2\sigma}$  are the control inputs, output of  $C_{1\sigma}$  are the errors to be kept small, and outputs of  $C_{2\sigma}$  are the output measurements provided to the controller.

The goal is to reduce the controller in the way that the closed-loop behavior is preserved as much as possible without sacrificing the stability of the original switched closed loop system. To do this, we develop generalized grammian framework for switched controller reduction which is inspired by the method in [9] which was proposed for controller reduction of linear dynamical systems.

The procedure is similar to generalized gramian framework for model reduction of switched system. In the first step one should find common generalized grammian for closed loop switched system. In other words we have to solve:

$$\begin{cases} \bar{A}_{\sigma}\bar{P}_{cg} + \bar{P}_{cg}\bar{A}_{\sigma}^* + \bar{B}_{\sigma}\bar{B}_{\sigma}^* < 0 \\ \forall \sigma \in K \end{cases} \quad (16)$$

and also

$$\begin{cases} \bar{A}_{\sigma}^*\bar{Q}_{og} + \bar{Q}_{og}\bar{A}_{\sigma} + \bar{C}_{\sigma}^*\bar{C}_{\sigma} < 0 \\ \forall \sigma \in K \end{cases} \quad (17)$$

to find the common generalized controllability and observability gramian respectively. Due to the fact that we are interested in the reduction of controller in our framework, we should find generalized gramians that have the following structure:

$$\bar{P}_{cg} = \text{diag}(P_1, P_2) > 0 \quad (18)$$

$$\bar{Q}_{og} = \text{diag}(Q_1, Q_2) > 0 \quad (19)$$

where  $P_2$  and  $Q_2$  are simultaneously balanceable and the are compatible to the order of the switched controller.

This can be done by solving Lyapunov linear matrix inequities using common LMI solvers.

The last step of the framework is to reduce the controller by balanced truncation of each sub-controllers based on  $P_2$  and  $Q_2$ .

#### B. Numerical Issues and the Algorithm

The last step of our framework is to apply balanced truncation on controllers. Balanced transformation can be ill-conditioned numerically when dealing with the systems with some nearly uncontrollable modes or some nearly unobservable modes. Difficulties associated with computation of the required balanced transformation in [11] draw some attentions to devise alternative numerical methods[12]. Balancing can be a badly conditioned even when some states are much more controllable than observable or vice versa. It is advisable then to reduce the system in the gramian based framework without balancing at all. Schur method and Square root algorithms provides projection matrices to apply balanced reduction without balanced transformation[1],[12]. This method can be easily applied to other gramian based method. In our generalized method for controller reduction we use the

same algorithm by putting generalized gramians  $P_2$  and  $Q_2$  into the algorithm instead of ordinary gramians. In order to improve the numerical algorithm we use Petrov-Galerkin projection to reduce switched controller.

Petrov-Galerkin projection for a dynamical system[1]:

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = g(x(t), u(t)) \end{cases}, \quad x \in \mathbb{R}^n \quad (20)$$

is defined as a projection  $\Pi = VW^*$ , where:  $W^*V = I_k$ ,  $V, W \in \mathbb{R}^{n \times k}$ ,  $k < n$ .

The reduced order model using this projection is:

$$\begin{cases} \dot{\hat{x}}(t) = W^* f(V\hat{x}(t), u(t)) \\ y(t) = g(V\hat{x}(t), u(t)) \end{cases}, \quad \hat{x} \in \mathbb{R}^k \quad (21)$$

In order to avoid numerical bad conditioning and also to increase the efficiency we use Schur or square root algorithm instead of balancing and directly Petrov-Galerkin projection matrices can be computed to reduce the switched controller.

### C. Stability, feasibility and Approximation Error

One of the important issues in model/controller reduction is preservation of the stability. In other words, the question is if the reduction technique method can preserve the stability of the original model in approximation. In the following proposition we show that the proposed framework for switched controller reduction is a stability preserving model reduction method that is, it preserves the stability of the original closed loop system under arbitrary switching.

**Proposition 2.** If the closed loop system described by (10)-(15) is stable, the closed-loop system with reduced switched controller resulting from the proposed algorithm is guaranteed to be quadratic stable.

**Proof:**

In the proposed method for original closed loop system, we have:

$$\begin{cases} \bar{A}_\sigma \bar{P}_{cg} + \bar{P}_{cg} \bar{A}_\sigma^* + \bar{B}_\sigma \bar{B}_\sigma^* < 0 \\ \forall \sigma \in K \end{cases}$$

where:  $\bar{P}_{cg} = \text{diag}(P_1, P_2) > 0$

Equivalently we have:

$$(A_\sigma + B_{2\sigma} L_\sigma D_{k\sigma} C_{2\sigma}) P_1 + P_1 (A_\sigma + B_{2\sigma} L_\sigma D_{k\sigma} C_{2\sigma})^* + (B_{1\sigma} + B_{2\sigma} L_\sigma D_{k\sigma} D_{21\sigma}) (B_{1\sigma} + B_{2\sigma} L_\sigma D_{k\sigma} D_{21\sigma})^* < 0 \quad (22)$$

and

$$(A_{k\sigma} + B_{k\sigma} F_\sigma D_{22\sigma} C_{k\sigma}) P_2 + P_2 (A_{k\sigma} + B_{k\sigma} F_\sigma D_{22\sigma} C_{k\sigma})^* + (B_{k\sigma} F_\sigma D_{21\sigma}) (B_{k\sigma} F_\sigma D_{21\sigma})^* < 0 \quad (23)$$

On the other hand, from our reduction framework for

switched controller using Petro-Galerkin projection we have:

$$W^*V = I_k, \quad V, W \in \mathbb{R}^{n \times k}, \quad k < n \quad (24)$$

$$K_r : (\hat{A}_{k\sigma} = W^* A_{k\sigma} V, \hat{B}_{k\sigma} = W^* B_{k\sigma}, \hat{C}_{k\sigma} = C_{k\sigma} V, \hat{D}_{k\sigma} = D_{k\sigma})$$

which  $K_r$  is projected switched controller (reduced order controller). The outcome of Square root algorithm for projection[1]:

$$\begin{aligned} P_2 W &= V \Sigma_1 \\ Q_2 V &= W \Sigma_1 \end{aligned} \quad (25)$$

where  $\Sigma_1 \in \mathbb{R}^{k \times k}$  is diagonal and positive definite. We know from (23):

$$(A_{k\sigma} + B_{k\sigma} F_\sigma D_{22\sigma} C_{k\sigma}) P_2 + P_2 (A_{k\sigma} + B_{k\sigma} F_\sigma D_{22\sigma} C_{k\sigma})^* < 0$$

which implies:

$$\begin{aligned} W^* ((A_{k\sigma} + B_{k\sigma} F_\sigma D_{22\sigma} C_{k\sigma}) P_2 + \\ P_2 (A_{k\sigma} + B_{k\sigma} F_\sigma D_{22\sigma} C_{k\sigma})^*) W < 0 \end{aligned} \quad (26)$$

On the other hand using (25) and then (24) we have,

$$\begin{aligned} W^* ((A_{k\sigma} + B_{k\sigma} F_\sigma D_{22\sigma} C_{k\sigma}) P_2 + P_2 (A_{k\sigma} + B_{k\sigma} F_\sigma D_{22\sigma} C_{k\sigma})^*) W \\ = W^* (A_{k\sigma} + B_{k\sigma} F_\sigma D_{22\sigma} C_{k\sigma}) P_2 W + W^* P_2 (A_{k\sigma} + B_{k\sigma} F_\sigma D_{22\sigma} C_{k\sigma})^* W \\ = W^* (A_{k\sigma} + B_{k\sigma} F_\sigma D_{22\sigma} C_{k\sigma}) V \Sigma_1 + \Sigma_1 V^* (A_{k\sigma} + B_{k\sigma} F_\sigma D_{22\sigma} C_{k\sigma})^* W \\ = (W^* A_{k\sigma} V + W^* B_{k\sigma} F_\sigma D_{22\sigma} C_{k\sigma} V) \Sigma_1 + \\ \Sigma_1 (W^* A_{k\sigma} V + W^* B_{k\sigma} F_\sigma D_{22\sigma} C_{k\sigma} V)^* \\ = (\hat{A}_{k\sigma} + \hat{B}_{k\sigma} F_\sigma D_{22\sigma} \hat{C}_{k\sigma}) \Sigma_1 + \Sigma_1 (\hat{A}_{k\sigma} + \hat{B}_{k\sigma} F_\sigma D_{22\sigma} \hat{C}_{k\sigma})^* \end{aligned}$$

Hence:

$$(\hat{A}_{k\sigma} + \hat{B}_{k\sigma} F_\sigma D_{22\sigma} \hat{C}_{k\sigma}) \Sigma_1 + \Sigma_1 (\hat{A}_{k\sigma} + \hat{B}_{k\sigma} F_\sigma D_{22\sigma} \hat{C}_{k\sigma})^* < 0 \quad (27)$$

where  $\Sigma_1 \in \mathbb{R}^{k \times k}$  is positive definite.

Similar to (11) for the closed-loop system with reduced switched controller we have:

$$\bar{A}_{r\sigma} = \begin{bmatrix} A_\sigma + B_{2\sigma} L_\sigma \hat{D}_{k\sigma} C_{2\sigma} & B_{2\sigma} L_\sigma \hat{C}_{k\sigma} \\ \hat{B}_{k\sigma} F_\sigma C_{2\sigma} & \hat{A}_{k\sigma} + \hat{B}_{k\sigma} F_\sigma D_{22\sigma} \hat{C}_{k\sigma} \end{bmatrix} \quad (28)$$

It is easy to see from (22) and (27) that for  $P = \text{diag}(P_1, \Sigma_1)$  we have:

$$\bar{A}_{r\sigma} P + \bar{A}_{r\sigma}^* P < 0 \quad (29)$$

Note that  $P$  is positive definite. Therefore  $x^* P x$  is the common quadratic Lyapunov function for the closed loop switched system with reduced controller.

In stability theory for switched system it is well-known sufficient condition for quadratic stability [13]. Hence, reduced order model is guaranteed to be quadratic stable.  $\square$

The same results hold, if we use balancing transformation instead of projection. The proof is straightforward and it is just based on the fact that for any matrix  $M \leq 0$ , all its leading square diagonal blocks are negative semidefinite.

As we can see, the presented framework for model reduction of switched system is stability preserving model reduction method. The error of approximation for each subsystem of the closed-loop switched system is bounded and it is given in terms of generalized Hankel singular values of the controller. This is the direct result of the theorem in [9] for linear controller reduction.

The system of LMIs in our framework is said to be feasible if a common generalized grammian exists. In general existence of a common lyapunov function is not guaranteed for switched systems [13], therefore we can not expect to have common generalized grammian for all linear switched controllers. One way to improve the feasibility of the proposed controller reduction method is to use recently proposed extended notion of generalized grammian which is called extended grammian [14].

#### IV. NUMERICAL EXAMPLE

In this section we have applied the proposed method for reduction of two switched linear controllers. The first example is a switched controller of order 5 and the second one is of order 20.

##### A. Fifth Order Switched controller:

We consider a randomly generated switched linear of the form (9) for which we have:

$$A_1 = \begin{bmatrix} -3.3428 & 0.7766 & -0.1894 & 0.5820 & 1.5424 \\ 0.7766 & -1.2319 & 0.3043 & 0.3098 & -0.2189 \\ -0.1894 & 0.3043 & -0.4807 & 0.0478 & 0.3770 \\ 0.5820 & 0.3098 & 0.0478 & -0.7472 & -0.6891 \\ 1.5424 & -0.2189 & 0.3770 & -0.6891 & -1.9965 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} -0.6905 & 0.8334 & -0.8545 & -1.316 & 0.1195 \\ -1.113 & -0.7869 & -1.917 & -0.455 & 1.335 \\ 0.748 & 1.968 & -0.7385 & -0.1609 & -0.0892 \\ -0.8108 & 0.5693 & 0.00355 & -4.924 & -2.485 \\ -0.8452 & -1.418 & 0.2512 & -2.278 & -1.872 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 & 0.5581 \\ 0.1024 & 0 \\ 1.8490 & -1.0816 \\ 1.1762 & 0.0374 \\ 0.2678 & -1.5963 \end{bmatrix}, B_2 = \begin{bmatrix} 1.543 & -0.3838 \\ -1.931 & -0.2474 \\ 0 & 0 \\ -0.3468 & -0.05512 \\ -0.1662 & -0.5688 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} -0.2914 & 0.0921 & -0.2622 & -0.4689 & -2.0424 \\ 0.8185 & -0.4314 & 0 & -1.2311 & 0 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} -1.78 & 0 & 0 & 0.1578 & -0.04066 \\ 0.9576 & 1.125 & 0 & 1.059 & 0 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} 0 & -0.2468 \\ 0.2342 & 0.9183 \end{bmatrix}, D_2 = \begin{bmatrix} 0 & 0.208 \\ 0.3414 & -1.024 \end{bmatrix}$$

A switched bimodal stabilizing H2 optimal controller  $K_\sigma(s) := (A_{k\sigma}, B_{k\sigma}, C_{k\sigma}, D_{k\sigma})$  is synthesized for the above switched system according to the Fig. 1. for which we have:

$$A_{k1} = \begin{bmatrix} -0.09566 & 0.1983 & 1.476 & 0.9568 & 9.094 \\ 4.643 & -2.1 & 2.024 & -0.07586 & 7.579 \\ 16.89 & -3.821 & 6.688 & -3.009 & 32.88 \\ 11.72 & -2.181 & 5.018 & -1.812 & 21.85 \\ -0.362 & -0.222 & -1.103 & -2.497 & -8.707 \end{bmatrix}$$

$$A_{k2} = \begin{bmatrix} 13.98 & -1.753 & -0.8545 & -5.245 & 0.5049 \\ 6.806 & -2.233 & -1.917 & -2.626 & 1.544 \\ 12.42 & 0.2864 & -0.7385 & -2.905 & 0.2101 \\ -3.915 & 0.9486 & 0.00355 & -4.263 & -2.563 \\ -9.733 & -0.8385 & 0.2512 & -0.9021 & -2.086 \end{bmatrix}$$

$$B_{k1} = \begin{bmatrix} -0.7196 \\ -1.371 \\ -6.89 \\ -3.92 \\ -0.5589 \end{bmatrix}, B_{k2} = \begin{bmatrix} 2.299 \\ 1.285 \\ 1.495 \\ -0.3372 \\ -0.5148 \end{bmatrix}, D_{k1} = D_{k2} = 0$$

$$C_{k1} = [2.181 \quad -0.2198 \quad 1.367 \quad 1.034 \quad 6.196]$$

$$C_{k2} = [8.599 \quad 0 \quad 0 \quad -0.7589 \quad 0.1955]$$

Fig. 2 shows the decay rate of the generalized Hankel singular values of the switched controller. It is clear from Fig. 2 that reduction of the controller to the fourth order switched controller should provide with accurate results.

The step response of the original closed loop system and closed loop system with reduced order controller of order 4 associated to randomly generate switching signal of Fig.3 is presented in Fig. 4.

We reduce the controller as much as possible i. e. to a first order switched controller. The step responses of the original closed loop system and the closed loop system with first order switched controller are shown in Fig. 5. These step responses are also associated to switching signal shown in Fig. 3.

According to Fig. 2 it was expected to have less accurate results in this case because too much input/output information are lost by omitting 4 states of the switched controller.

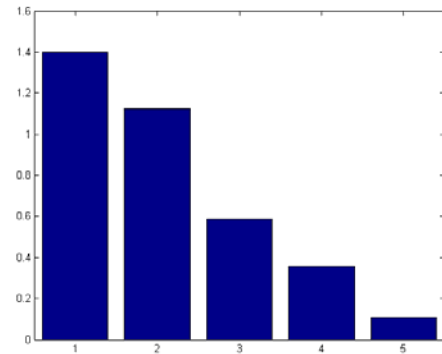


Fig. 2. Generalized Hankel Singular Value( $\gamma_i$ )

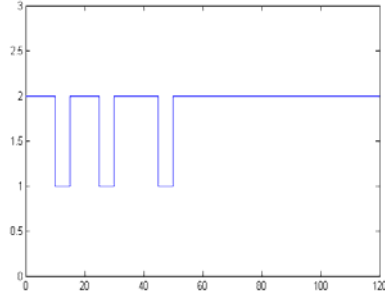


Fig. 3. Randomly generated switching signal

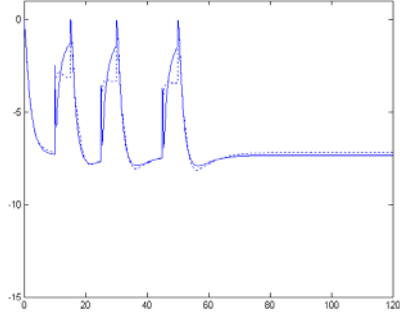


Fig. 4. Step response of original closed loop system (solid line) and the closed loop system with the reduced switched controller of order 4 (dotted).

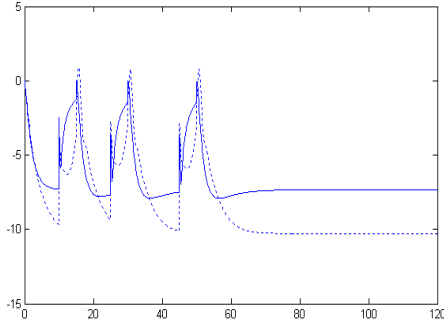


Fig. 5. Step response of original closed loop system (solid line) and the closed loop system with the reduced first order switched controller (dotted).

#### B. Bimodal Switched controller of order 20:

We consider a randomly generated bimodal switched linear system of order 20. Similar to the previous example, we designed a switched bimodal stabilizing  $H_2$  optimal controller for the system. The switched controller is of order 20 with the generalized Hankel singular values which are shown in Fig. 6. It is clear from Fig. 6 that most of the input/output behavior information are embedded in the first two states of the controller. We expect that reduction of the controller to the second order switched controller should provide us with accurate results.

The step response of the original closed loop system and closed loop system with reduced second order switched controller associated to randomly generate switching signal of Fig. 7 is presented in Fig. 8.

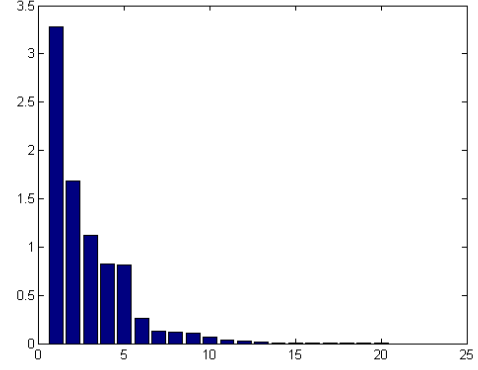


Fig. 6. Generalized Hankel Singular Value ( $\gamma_i$ )

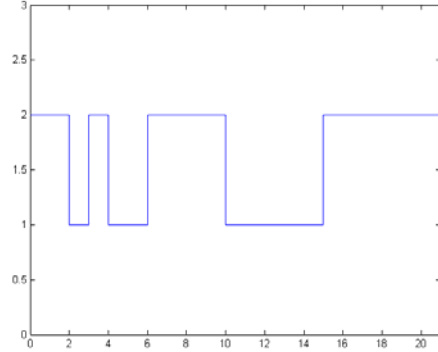


Fig. 7. Randomly generated switching signal

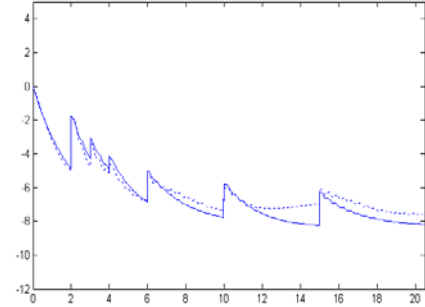


Fig. 8. Step response of original closed loop system (solid line) and the closed loop system with the reduced second order switched controller (dotted).

#### V. CONCLUSION

A method for switched controller order reduction is presented in this paper. The proposed method is based on the generalized gramian framework for model reduction which needs to solve LMI's in the reduction procedure.

This method preserves the stability of the original closed loop switched system under arbitrary switching signal and is applicable to both continuous and discrete time systems. It is also a general method meaning that different gramian based reduction method can be developed in this framework for switched controller reduction. The presented approach

due to the fact that uses common generalized gramian not only preserves the stability but also it reduces the sub-controllers in one shot using global projection matrices.

One of the drawbacks of the method is that it is not guaranteed to be feasible because it is not always possible to find a common Lyapunov function for switched systems. Error is bounded but it is not guaranteed to be always small enough. There are different directions for further extensions such as using optimization, piecewise gramians and also various generalized gramians.

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